

GLOBAL PROPERTIES OF SPHERICAL CURVES

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Let α be a closed curve regularly embedded in Euclidean three-space satisfying suitable differentiability conditions. In addition, suppose α is nonsingular, i.e., free of multiple points. In 1968, B. Segre [4] proved the following about such curves.

Theorem. *If α is nonsingular and lies on a sphere, and O denotes any point of the convex hull of α with the condition that O (if lying on α) is not a vertex of α , then there are always at least four points of α whose osculating plane at each of those points passes through O . If O is a vertex of α then there are at least three points of α whose osculating plane at each of those points passes through O .*

All terms used in the statement of the theorem are defined later in this paper.

To quote H. W. Guggenheimer [2] who reviewed [4], "The 12-page proof is rather complicated." Here we present a shorter and hopefully more transparent proof of this theorem. In addition, we need only require that the spherical curve α be of class C^2 whereas Segre's proof requires α be of class C^3 . Also, we obtain, with no extra effort, a similar theorem which holds if α 's only singularity is one double point; in this case, the above mentioned minimums must be reduced by two.

In the last section of this paper we characterize spherical curves with the following property: for every point O of the convex hull of α , other than a vertex of α , there exists the same (necessarily even) number of distinct points of α whose osculating plane at each of those points passes through O .

The proofs of many results in this paper ultimately depend on ideas contained in a paper by W. Fenchel [1].

Throughout this paper we use the following conventions. By a curve we mean a regular C^2 function $\alpha: D \rightarrow E^3$, where D is an interval (with or without end points) or a circle, and E^3 is Euclidean three-space. We let α denote both the function and its configuration $\alpha(D)$ in E^3 . When D is a circle we say α is closed. If D is a closed interval we may sometimes refer to α as an arc. We say a point P in E^3 is a multiple point of α if it is the image of $k > 1$ points of D . If $k = 2$ then P is called a double point. At a multiple point P we will think of P as k distinct points each traversed once by α as we traverse D once. If α has no multiple points, then we say α is nonsingular.

1. Geodesic curvature

Let α be an oriented spherical curve; i.e., α lies on a sphere S in E^3 and has a preferred direction of traversal. Let S be oriented, say, with respect to the outward pointing normal. We denote by k the geodesic curvature of α as a curve in S . It is defined by $k = (d^2\alpha/ds^2) \cdot n$, where s is the arc length parameter of α consistent with its orientation, and n is $d\alpha/ds$ rotated $+90^\circ$ in the tangent plane to S at its point of contact with S . Since α is C^2 , k is a continuous function on α .

At each point P of α there is in S a circle tangent to α which best approximates α near P . This circle $\omega(P)$ is the osculating circle to α at P ; it is easy to see that $\omega(P)$ is the intersection of the sphere S and the osculating plane $\pi(P)$ to α at P , when α is viewed as a curve in E^3 . We have the following obvious lemma.

Lemma 1. *Let α be a spherical curve and $P \in \alpha$. Then $k(P) = 0$ if and only if $\pi(P)$ goes through the center of S .*

We will need some lemmas about spherical curves proved by Fenchel [1]. Actually we state mild generalizations of these lemmas; see [1], [5] for their proofs. In these lemmas we speak of a set on the sphere being to the left of a curve. By this we mean that when the tangent vector to the curve in the preferred direction is rotated $+90^\circ$ it points into the set. Also when we say a point P is between points A and B we mean that either A and B are antipodes or if A and B are not antipodes then P lies on the shorter geodesic arc through A and B .

Lemma 2. *A nonsingular spherical curve α with $k \geq 0$ and not identically zero connects two points A and B of a great circle γ without otherwise meeting it. Then A and B are not antipodes of one another. In addition the region bounded by the curve and the smaller great circular arc AB of γ and lying in a hemisphere is to the curve's left.*

Lemma 3. *Let α be a nonsingular spherical curve with $k \geq 0$, and let γ be an arbitrary great circle which meets α in at least two points. Then there is a subarc α_γ of α with the following characteristics:*

1. *The end points A and B of α_γ lie on γ .*
2. *α_γ has otherwise no points in common with γ .*
3. *All other points of intersection of α with γ lie between A and B .*

Remark. If α_γ contains a point P for which $k(P) > 0$, then A and B are not antipodal by Lemma 2. In particular, more than a half circle of γ is free of points of intersection with α .

2. Fenchel's theorem

The convex hull of a point set M in Euclidean space is the smallest convex set containing M . Let Ω be the convex hull of a spherical curve α . The next lemma characterizes the points of Ω ; for its proof see [1, Satz A].

Lemma 4. *For O to be an element of Ω it is necessary and sufficient that there exists a plane λ through O such that O is in the convex hull of $\alpha \cap \lambda$.*

Throughout this section we take O to be the center of the sphere S on which α lies. With this choice for O , Lemmas 3 and 4 lead immediately to a theorem due to Fenchel [1, Satz II']. This theorem is restated to include the possibility that O is an element of the boundary of Ω as well as the interior of Ω .

Theorem 1 (Fenchel). *Suppose α is closed and nonsingular except perhaps for one double point. If $O \in \Omega$, and α does not contain a great semicircular arc, then the geodesic curvature of α changes sign at least twice.*

The same lemmas can be used to prove the following extension of Theorem 1. This will be shown here.

Theorem 2. *Suppose α is closed and nonsingular. If $O \in \Omega$, and α does not contain a great semicircular arc, then the geodesic curvature of α changes sign at least four times.*

Remark. It is easy to construct examples of closed nonsingular spherical curves whose geodesic curvature changes sign only twice and which necessarily contain a great semicircular arc. It is a consequence of Lemma 2 that these curves lie in a hemisphere determined by the great semicircular arc.

The remainder of this section is devoted to a proof of Theorem 2. Before we proceed we introduce some notation. If α is a non-closed spherical curve, and P, Q are two points of α , then by $P\alpha Q$ we mean the oriented arc running along α from P to Q . If P, Q are two points of the sphere S which are not antipodal, then PQ denotes the smaller great circular arc through P and Q oriented from P towards Q . To denote the larger great circular arc connecting P and Q , we write PAQ where A is on the great circle through P and Q but $A \notin PQ$. By a Jordan curve we mean a nonsingular continuous image of a circle.

Proof of Theorem 2. Let α be a closed nonsingular curve lying on a sphere S with center O , and suppose that α contains no great semicircular arc. In particular, α 's geodesic curvature k is not identically zero. Also suppose $O \in \Omega$, the convex hull of α . By Theorem 1 we already know that k changes sign at least twice. We will show that the supposition that k changes sign only twice leads to a contradiction. Therefore suppose k changes sign twice at the points A and B of α . Let α^1 and α^2 be the two curves into which α is separated by A and B , both oriented so that their geodesic curvature is nonnegative (and, of course, not identically zero). Suppose α^1 and α^2 begin at A and end at B .

By Lemma 2 there is a plane λ through O such that O is in the convex hull of $\lambda \cap \alpha$. Let $\gamma = \lambda \cap S$; it is, of course, a great circle. There are two cases to consider. Either

1. α meets γ in at least three points and these points do not lie in an open half circle of γ , or
2. α meets γ in two points, which are necessarily antipodal.

Case 1. Let C, D, E be distinct points at which $\alpha = \alpha^1 \cup \alpha^2$ meets γ and which do not lie in an open half circle of γ . We may suppose that C and D are points of α^1 ; in fact, suppose C precedes D in α^1 . Since α^1 meets γ in at least two points, Lemma 3 implies that there exists a subarc α^1_7 with the characteristics 1, 2, and 3 of that lemma. Also α^1_7 is not a great semicircular arc. The remark following Lemma 3 implies that E must be a point of α^2 . We may assume that C and D are the end points of α^1_7 ; if the new C, D, E lie in an open half circle of γ so do the old C, D, E .

Let H be the closed hemisphere determined by γ and not containing α^1_7 except for the end points C and D . Let L be the region to the left of the oriented Jordan curve $C\alpha^1_7D \cup DC$ together with its boundary. Lemma 3 implies that $\alpha^1 \subset H \cup L$. In particular $A, B \in H \cup L$; hence α^2 must begin and end in $H \cup L$. The boundary of $H \cup L$ is the Jordan curve $\alpha^1_7 \cup DEC$. Now if α^2 is not contained in $H \cup L$, it must cross the boundary along DEC (excluding the end points D and C). Remember that α^1 and α^2 meet only at A and B . We assume without loss of generality that α^2 crosses DEC . If α^2 did not cross DEC , then it would be tangent to γ at E . We could then rotate λ a bit about the diameter of S through C or D so that α crosses γ at points which we still call C, D, E and which still do not lie in an open half circle of γ . Since α^2 meets γ at least twice, Lemma 3 implies the existence of a subarc α^2_7 . Let α^2_7 begin at F and terminate at G . Characteristic 3 of α^2_7 implies that at least one of the points F and G is not between C and D . At this stage of the argument we suppose that F does not lie between C and D . The argument is similar if we suppose that G does not lie between C and D .

Consider the oriented Jordan curve $A\alpha^1D \cup DF \cup F\alpha^2A$. If D and F are antipodal, then here DF is the half great circle not containing G ; see Fig. 1.

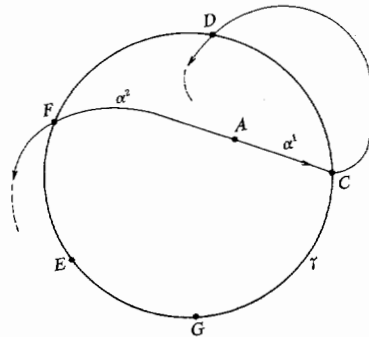


Fig. 1

Note that $D\alpha^1B$ and $F\alpha^2B$ cannot cross the Jordan curve. That $F\alpha^2B$ does not cross DF is the only part of the preceding statement which may not be im-

mediately clear. However $F\alpha^2B$ may only cross γ along FG which is less than a half circle; also DF is at most a half circle. Thus DF meets FG only at F . Thus $F\alpha^2B$ meets DF only at F . Now $D\alpha^1B$ and $F\alpha^2B$ are on opposite sides of the Jordan curve near D and F , respectively. This is clear since α^1 is entering H at D and α^2 is leaving H at F . Thus B is both to the right and the left of the Jordan curve, which is a contradiction.

Case 2. Let C and D be the two points in which α meets γ . As already noted C and D are necessarily antipodal. This case can be reduced to Case 1 since there must be a great circle through C and D which intersects α at a third point E . Clearly C, D, E do not lie in an open half circle.

Remark. We do not use the fact that α^1 and α^2 join at A and B in a C^2 fashion, but only that they begin and end at A and B , respectively.

3. Segre's theorem

Generally, if P is a point of a curve α then at P α passes through the osculating plane to α at P . However if this does not happen we call P a vertex of α . Thus by a vertex of a curve α we mean a point P of α with the property that near P α lies on one side of the osculating plane to α at P .

Theorem 3. *Let α be a closed curve on the sphere S and let $0 \in \Omega$, α 's convex hull. Then*

- (i) *if α is nonsingular and 0 is not a vertex of α , there exist at least four points of α whose osculating plane at each of those points passes through 0 ,*
- (ii) *if α is nonsingular and 0 is a vertex of α , there exist at least three points of α whose osculating plane at each of those points passes through 0 ,*
- (iii) *if α 's only singularity is one double point and 0 is not a vertex of α , there exist at least two points of α whose osculating plane at each of those points passes through 0 .*

The idea behind the proof lies in the observation that Theorem 3 follows trivially from Theorems 1 and 2 by means of Lemma 1 if 0 is the center of S . So if 0 is not the center of S we let α^* be the projection of α into a sphere Σ centered at 0 and apply Theorems 1 and 2 to α^* to get the required number of points of α^* whose osculating plane at each of those points passes through 0 . If $0 \in \alpha$, then α^* is not a closed curve but one can still show that α^* has the required number of points whose osculating plane at each of those points passes through 0 . Finally we observe by Lemma 5 that an osculating plane at a point of α^* passes through 0 if and only if the osculating plane at the corresponding point of α does so.

We now introduce the notation which will be used in the proofs of Lemma 5 and Theorem 3. Let α be a closed curve on S , and Ω the convex hull of α . Suppose that 0 is any element of Ω and Σ is a sphere centered at 0 . Let $p: S \rightarrow \Sigma$ be the projection of S into Σ through 0 . When $0 \in \alpha$, p is understood to be defined only on $S - \{0\}$. Denote the image of $P \in S$ under $p: S \rightarrow \Sigma$ by P^* .

If 0 is in interior of S , we let α^* denote the image of α under p . If $0 \in \alpha$, note first that $p(\alpha)$ is contained in a hemisphere H with boundary γ^* , where γ^* is the intersection of the tangent plane to S at 0 with Σ . Assume 0 is not a multiple point of α ; then the limits of P^* as P approaches 0 along α first from one side and then the other are two antipodal points on γ^* . We adjoin these points to $p(\alpha)$ and denote the resulting arc by α^* . When 0 is a multiple point of α , we adjoin points of γ^* to $p(\alpha)$ as above to get a collection of arcs denoted by α^* . Then let Ω^* be the convex hull of α^* . Let $\pi(P)$ and $\pi^*(P^*)$ denote the osculating planes to α at P and α^* at P^* , respectively.

Lemma 5. *Suppose $P \neq 0$. Then $\pi(P)$ passes through 0 if and only if $\pi^*(P^*)$ goes through 0. Moreover, if $\pi(P)$ passes through 0, then P is a vertex of α if and only if P^* is a vertex of α^* .*

Proof. The projection $p: S \rightarrow \Sigma$ is a C^∞ diffeomorphism of S onto its image. Thus the order of contact between two curves on S and their images under p on Σ is preserved (except if the contact is at $0 \in \alpha$).

Let $\omega(P)$ and $\omega^*(P^*)$ denote the osculating circles to α at P and α^* at P^* , respectively. Suppose $\pi(P)$ passes through 0. Since $\omega(P)$ lies in $\pi(P)$ which passes through 0, its image under p is a (great) circle on Σ if $0 \notin \alpha$ and is a half (great) circle on Σ if $0 \in \alpha$. Let $\omega(P)^*$ denote the circle in which $p(\omega(P))$ lies on Σ . Since the order of contact is preserved, $\omega(P)^* = \omega^*(P^*)$. Thus both $\pi(P)$ and $\pi^*(P^*)$ contain $\omega(P)^*$. Hence $\pi(P) = \pi^*(P^*)$ passes through 0. The converse is proved in an identical fashion.

Now suppose $\pi(P)$ passes through 0. Then, by the above, $\pi(P) = \pi^*(P^*)$. If α lies on one side of $\pi(P)$ near P , clearly α^* lies on one side of $\pi^*(P^*)$ near P^* and conversely. That is, P is a vertex of α if and only if P^* is a vertex of α^* .

Proof of Theorem 3. We separate the proof into two cases according as $0 \in \alpha$ or not.

Suppose $0 \notin \alpha$. Then it is clear that $0 \in \Omega^*$ since $0 \in \Omega$. Thus we may apply Theorems 1 and 2 to α^* lying on Σ . If α is nonsingular, so is α^* ; thus α^* has at least four points where its geodesic curvature is zero. If α has just one double point, so does α^* ; thus α^* has at least two points where its geodesic curvature is zero. By Lemma 1, at each of these points of α^* the osculating plane passes through 0. Hence by Lemma 5 the osculating planes at the corresponding points of α pass through 0. Thus we have proved (i) and (iii) for the case $0 \notin \alpha$.

Suppose $0 \in \alpha$ and 0 is not a multiple point of α . Assume now α is oriented. By means of p we orient α^* . Denote the beginning of α^* by A and the end by B . Let ω be the osculating circle to α at 0. Its image under p including end points, denoted by ω^* , is a half great circular arc of Σ . It is easy to see that ω^* also begins at A and ends at B . Also ω^* and α^* are tangent at A and B . If 0 is not a vertex of α , then α^* is on opposite sides of ω^* in H near A and B ; see Fig. 2. If 0 is a vertex of α , then α^* is on the same side of ω^* in H

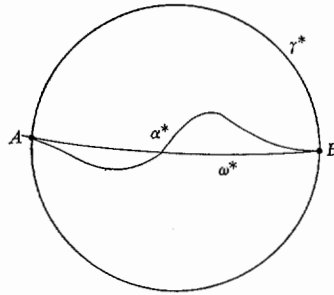


Fig. 2

near A and B . Let k^* be the geodesic curvature of α^* . Then using Lemma 2 and the idea of parity, one can show the following hold :

1. k^* changes sign at least twice if 0 is not a vertex of α and α is nonsingular,
2. k^* changes sign at least twice if 0 is a vertex of α and α is nonsingular,
3. k^* changes sign at least once if 0 is not a vertex of α and α 's only singularity is one double point.

Again apply Lemmas 1 and 5, in that order, to prove (i), (ii), and (iii) for the case where $0 \in \alpha$ and 0 not a multiple point of α . If 0 is the double point of α the proof of (iii) is immediate.

Corollary. *Let α be a C^3 closed nonplanar curve in E^3 with no pair of directly parallel tangents. Then α has at least four vertices.*

For the proof of this corollary see Segre [4, p. 263] where the same result is proven for C^4 curves. Our results allow his proof to go through for C^3 curves. Actually the corollary follows immediately from Theorem 2 and the remark following Theorem 2 since the tangent indicatrix of a nonplanar curve cannot lie in a hemisphere.

4. A characterization

In this section we find a characterization for a (possibly singular) closed curve α lying on the sphere S and having the property that for each point 0 in its convex hull Ω except for vertices of α there exists the same (necessarily even) number of distinct points of α whose osculating plane at each of those points passes through 0 .

The next lemma is especially important in this section. It follows by means of stereographic projection from a similar fact for plane curves due to Kneser ; see [3, p. 48] for Kneser's theorem and its proof. When we say that the circle ω lies between the (disjoint) circles ω^1 and ω^2 on the sphere S we mean that ω is in the connected component of $S - (\omega^1 \cup \omega^2)$ whose boundary is $\omega^1 \cup \omega^2$.

Lemma 6. *Let α be spherical arc with monotone geodesic curvature k . Let $P, Q,$ and R be three points of α with Q between P and R . Then $\omega(Q)$ is be-*

tween $\omega(P)$ and $\omega(R)$ if it is not equal to $\omega(P)$ or $\omega(R)$. Moreover, $\omega(Q) = \omega(P)$ (respectively, $\omega(R)$) only if $k(Q) = k(P)$ (respectively, $k(R)$).

At this point we make some additional assumptions about the closed spherical curve α which will hold throughout the remainder of this section. First, we require that there exists at most a finite number of points of α at which the geodesic curvature k takes on an extreme value. This is equivalent to requiring that α has at most a finite number of vertices since the vertices of α occur at the extremes of k . Secondly, we assume k is strictly monotone between the vertices of α . This second condition rules out the possibility of α having an arc of points with the same osculating plane.

Let B denote the closed ball whose boundary S contains the closed curve α . Clearly $\Omega \subset B$.

Theorem 4. *Suppose α has n vertices. If $0 \in B$, then there exist at most n points of α whose osculating plane at each of those points passes through 0 .*

Proof. Let V_1, V_2, \dots, V_n denote the vertices of α as they occur in making one circuit of α . Using the notation of § 2, we set $\alpha^i = V_i\alpha V_{i+1}$ for $i = 1, 2, \dots, n$, where $V_{n+1} = V_1$. We will show for each integer i , where $1 \leq i \leq n$, there exists at most one point $P \in \alpha^i$ such that $0 \in \pi(P)$. This immediately implies the theorem.

Suppose, to the contrary, that α^i contains two points P and Q such that $0 \in \pi(P) \cap \pi(Q)$. In particular, $\pi(P) \cap \pi(Q) \neq \emptyset$; hence $\omega(P) \cap \omega(Q) \neq \emptyset$. This is impossible by Lemma 6 since k is strictly monotone on α^i .

Remark. Note that $V_i \in \alpha^{i-1} \cap \alpha^i$ for $i = 1, 2, \dots, n$, where $\alpha^0 = \alpha^n$. Hence if $0 \in B$ and, in addition, $0 \in \pi(V_i)$, then there exist strictly less than n points of α whose osculating plane at each of those points passes through 0 .

Corollary. *Suppose α has n vertices. If $0 \in \Omega$, then there exist at most n points of α whose osculating plane at each of those points passes through 0 .*

Let V_1, V_2, \dots, V_n be the vertices of α . Note that n is necessarily even since it is the number of extreme points of the geodesic curvature of α .

Theorem 5. *Suppose $\omega(V_i) \cap \alpha = \{V_i\}$ for $i = 1, 2, \dots, n$. Then for every $0 \in \Omega - \{V_1, V_2, \dots, V_n\}$ there exist exactly n points P_1, P_2, \dots, P_n of α such that $0 \in \pi(P_i)$ for $i = 1, 2, \dots, n$, and conversely.*

Proof. Let $B' = B - \bigcup_{i=1}^n \pi(V_i)$. Also let B'_m be the set of points 0 in B' with the property that there exist exactly m points P_1, P_2, \dots, P_m of α such that $0 \in \pi(P_i)$ for $i = 1, 2, \dots, m$.

Let $\Omega' = \Omega - \{V_1, V_2, \dots, V_n\}$. For $i = 1, 2, \dots, n$, the assumption $\omega(V_i) \cap \alpha = \{V_i\}$ implies $\Omega \cap \pi(V_i) = \{V_i\}$. Thus Ω' is a connected subset of B' . The theorem is proved by showing that for any nonnegative integer m , B'_m is an open and closed subset of B' . This implies $\Omega' \subset B'_m$ for some nonnegative integer m . Then we show $m = n$.

The fact that B'_m is both open and closed in B' follows in three steps:

Step 1. $B'_m \subset \text{interior } \bigcup_{m \leq j} B'_j$. Let $0 \in B'$ and suppose there exist m points P_1, P_2, \dots, P_m of α such that $0 \in \pi(P_i)$ and P_i is not a vertex of α for

$i = 1, 2, \dots, m$. We will show for each integer i , where $1 \leq i \leq m$, there exists a neighborhood N_i of P_i in α with the property that $U_i = \bigcup_{P \in N_i} \pi(P) \cap B'$ is an open set of B' containing 0. Moreover, we may assume N_1, N_2, \dots, N_m are mutually disjoint. It is then clear that $U = \bigcap_{i=1}^m U_i$ is a neighborhood of 0 in $\bigcup_{m \leq j} B'_j$.

Consider the point P_i . Since P_i is not a vertex there exists an open neighborhood N_i of P_i in α on which k is strictly monotone. By Lemma 6, N_i does not contain P_j , where $j \neq i$. Let P'_i and P''_i be the boundary points of N_i . It follows from Lemma 6 that $\bigcup_{P \in N_i} \omega(P)$ is an open set of S ; it is the component of $S - [\omega(P'_i) \cup \omega(P''_i)]$ containing P_i . Then $U_i = \bigcup_{P \in N_i} \pi(P) \cap B'$ is an open set of B' . In fact U_i is the component of $B' - [\pi(P'_i) \cup \pi(P''_i)]$ containing P_i . Clearly $0 \in U_i$ since $P_i \in N_i$.

Step 2. B'_m is closed in B' . Let $0_i, i = 1, 2, \dots$, be a sequence of points in B'_m approaching $0 \in B'$. Thus for each $i = 1, 2, \dots$, there exist exactly m points $P_{i1}, P_{i2}, \dots, P_{im}$ of α such that $0_i \in \pi(P_{ij})$ for $j = 1, 2, \dots, m$. By taking subsequences if necessary, we may assume that P_{ij} approaches a point P_j as i approaches infinity for $j = 1, 2, \dots, m$. By continuity $0 \in \pi(P_j)$ for $j = 1, 2, \dots, m$. Thus there are at least m points of α whose osculating plane at each of those points passes through 0 unless $P_j = P_k$ for some $j \neq k$. Suppose this; then in any neighborhood of $P_j = P_k$ there exist the distinct points P_{ij}, P_{ik} , for i sufficiently large. Since $0_i \in \pi(P_{ij}) \cap \pi(P_{ik})$, $\omega(P_{ij}) \cap \omega(P_{ik}) \neq \emptyset$. By Lemma 6, $P_j = P_k$ is a vertex of α . But this contradicts the assumption $0 \notin \bigcup_{i=1}^n \pi(V_i)$. Thus $P_j \neq P_k$ for all $j \neq k$ between 1 and m inclusive. By Step 1 there exist at most m points P_1, P_2, \dots, P_m of α with $0 \in \pi(P_j)$.

Step 3. B'_m is open in B' . This step follows immediately from Step 1 and Step 2 since $B'_m = \emptyset$ for $m > n$ by Theorem 4.

We now know that $\Omega' \subset B'_m$ where $m \leq n$. Suppose $m < n$. We will show this leads to a contradiction. Let $0 \in \alpha \cap \Omega'$. Since $0 \in \Omega'$, there exist m points P_1, P_2, \dots, P_m with $0 \in \pi(P_i)$ for $i = 1, 2, \dots, m$. In the notation of the proof of Theorem 4, there exists an arc α^i for some integer between 1 and n inclusive with the following property: there exists no point $Q \in \alpha^i$ such that $0 \in \pi(Q)$. Thus $\omega(V_i)$ and $\omega(V_{i+1})$ do not have 0 between them. Hence, say, $\omega(V_i)$ and 0 are separated by $\omega(V_{i+1})$. In particular V_i and 0 are on opposite sides of $\omega(V_{i+1})$. Thus α must meet $\omega(V_{i+1})$ at points other than V_{i+1} .

The converse follows from the remark following the proof of Theorem 4.

q.e.d.

It may still be that for every point 0 of Ω' there exists the same number of points of α whose osculating plane at each of those points passes through 0 even though $\omega(V_i) \cap \alpha \neq \{V_i\}$ for some integer $i, 1 \leq i \leq n$. For this to happen the following must be true: if, say, V_1 is a vertex of α and $\omega(V_1)$ intersects α in more than V_1 , then there must be another vertex V_i for some integer $i, 2 \leq i \leq n$, such that $\pi(V_i) = \pi(V_1)$. Also, for points P near V_1 and Q near $V_i, \pi(P)$ and $\pi(Q)$ must be on opposite sides of $\pi(V_1) = \pi(V_i)$.

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